Effect of delayed differentiation on a multiproduct vendor-buyer integrated inventory system with rework

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**ABSTRACT**

This study explores the effect of delayed differentiation on a multiproduct vendor-buyer integrated inventory system with rework to identify its potential benefits and provide managers with in-depth information for operational decision-making. The main considerations of the proposed study include a multiproduct fabrication plan to increase machine utilization, a rework process to ensure product quality, and a multi-shipment policy to distribute the end products. In addition, these products sharing an intermediate part for which a two-stage fabrication scheme is adopted, wherein the common parts are produced at the first stage and the end products are manufactured at the second stage. The aim is to reduce the overall system costs and shorten the replenishment cycle time. Mathematical modeling and optimization methods were employed to derive the closed-form optimal replenishment cycle time and delivery decisions. We demonstrated the applicability of our research results through numerical examples and revealed that for both linear and nonlinear relationships between the common intermediate part’s completion rate $a$ and its practical value at $a$, our proposed two-stage production scheme with delayed differentiation is considerably beneficial vis-à-vis single-stage schemes in saving overall system costs and reducing the replenishment cycle time.

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1. Introduction

Conventional economic production quantity (EPQ) model considers a single product fabrication with all items produced are of perfect quality and customer’s demand satisfied by a continuous inventory issuing policy [1–3]. However, in real world supply chain systems, vendors usually adopt a multi-product production plan to get the most out of machine utilization and consider reworking of nonconforming items to lower their production cost. Aggarwal [4] presented a simple grouping idea under a common order cycle to resolve the multi-product inventory system. A computation procedure was also presented to derive optimal values of common order cycle. Rosenblatt and Rothblum [5] studied the multi-item inventory systems under a single-resource capacity constraint. Two solution procedures were proposed to derive optimal capacity policy. A numerical example is used to show that their solution procedures can be applied to different types of cost functions. Aliyu and Andijani [6] examined a multi-item production-inventory system with shortages, deterministic demand, deterioration, and capacity and budget constraints. Linear quadratic theory was used to solve the optimal control policy. Balkhi and Foul [7] studied a multi-product inventory model with deterministic demand, production, and deterioration rates for each product in finite time periods. Shortage is allowed and backordered
for each product. For each product, they derived the optimal production and restarting times in each period that minimize the total inventory costs. Rahmani et al. [8] investigated a two-stage real capacitated production system with uncertain demand and production costs. A mixed-integer programming model was developed to the problem. An initial robust schedule was obtained and it can be improved against any possible occurrences of uncertain parameters. They provided a real case to demonstrate the practical use of their model. Chiu et al. [9] developed an exact mathematical model to simultaneously derive the production and shipment decisions for a multi-product inventory system with a rework process. A single-stage production process is considered without involving the common intermediate part. Their results enable managers of such a specific system to better understand and control over the effects of variations in different system parameters on the optimal production-shipment policy and on the expected system costs. Additional studies related to the multi-product inventory systems can also be found elsewhere [10-14].

In multi-item production planning, if multiple products share a common intermediate part, vendors would always be interested in evaluating a two-stage fabrication scheme with the first stage making common intermediate parts for all products, and the second stage producing the end products to reduce overall system costs and shorten the replenishment cycle time. Gerchak et al. [15] developed a model for an arbitrary number of products with general joint demand distribution. They discussed the case of using a service-level measure where rationing of common components might be required and characterized the implied rationing rule. Garg and Tang [16] stated that practically most product families have a number of points of differentiation. They developed two models to investigate products with more than one point of differentiation. Benefits of delayed differentiation at each point in each model are examined. Necessary conditions are decided when one type of delayed differentiation is more beneficial than the other. They found that variations in demand and lead times have significant effects on determining which point of differentiation should be delayed. Graman [17] developed a two-product, single-period, order-up-to-cost model to assist in deciding the inventory levels of end products and postponement capacity. A non-linear programming was used to determine the optimal solutions to inventory levels and capacity that minimize the total system cost. He indicated that altering product value, holding cost, cost of postponement, packaging cost, and fill rate can reduce expected total cost and increase postponement capacity. Other studies addressed various aspects of multi-product systems with delayed product differentiation can also be found elsewhere [18-21]. Also, in real manufacturing environments, due to various uncontrollable factors during production process, generation of defective items is inevitable. Quality assurances, such as inspection of product quality, rework of all repairable items, and scrapped of defective items, have been extensively studied in past decades [22-28]. Also, in contrast to the assumption of continuous issuing policy in conventional EPQ model, most nowadays supply chain systems practically adopt a periodic multi-shipment policy to distribute end products to their customers. Studies of various aspects of periodic or multi-delivery issues of vendor-buyer integrated systems have been extensively carried out during past decades [29-44].

Inspired by the potential benefits derived from applying delayed differentiation to multi-product systems, and seeking to provide managers of transnational enterprises with information to assist them in achieving the key operational goals such as maximizing machine utilization, ensuring product quality, lowering overall operating costs, and shortening response time, this study extends a prior work [9] and explores the effect of delayed differentiation on a multi-product vendor-buyer integrated inventory system with rework. Since little attention has been paid to this specific research area, the present study is intended to bridge the gap.

2. Model description and mathematical analysis

Description of the proposed multi-product vendor–buyer integrated inventory system with delayed differentiation strategy and rework using a single-machine production scheme is as follows. Consider a vendor has annual demand \(\lambda_i\) for \(L\) different products (where \(i = 1, 2, ..., L\)) that must be satisfied. These \(L\) customized end items share a common intermediate part and are
manufactured using a two-stage process. The first stage produces only the common intermediate components, and the second stage fabricates in sequence $L$ different customized end products under the common production cycle time policy. The objectives of the proposed production plan are to maximize machine utilization, shorten the replenishment cycle time, and minimize total production-inventory-delivery costs. The common intermediate part is manufactured at a rate of $P_{1,0}$ in stage one. After that, $L$ different customized end products are produced in order under a common cycle time policy in stage two (see Fig. 1), at a rate of $P_{1,i}$.

All items made are screened and unit inspection cost is included in unit production cost $C_i$. The production processes in each stage (either for common intermediate part or for customized end products) may randomly produce $x_i$ portion of defective items at a rate of $d_{1,i}$ and $d_{1,i} = P_{1,i} x_i$ (where $i = 0, 1, 2, \ldots, L$; with $i = 0$ denotes that it is for the production of common intermediate part in the stage 1). Under the ordinary assumption of the EPQ model without shortages, the constant production rate $P_{1,i}$ must be larger than the sum of demand rate $\lambda_i$ and production rate of defective items $d_{1,i}$. That is: $(P_{1,i} - d_{1,i} - \lambda_i) > 0$ or $(1 - x_i - \lambda_i/P_{1,i}) > 0$. It is further assumed that all defective items can be reworked and repaired. The rework processes starts immediately after the end of regular production processes in each production cycle (see Fig. 2), at a rate of $P_{2,i}$.

**Fig. 1** Inventory level of perfect quality common intermediate parts and customized final products in the proposed two-stage multi-product vendor-buyer integrated inventory system with rework

**Fig. 2** Inventory level of defective items in both stages of the proposed two-stage multi-product system
Fig. 3 Inventory level of common intermediate parts waiting to be fabricated into customized final products in the stage 2 of the proposed two-stage multi-product system

Upon completion of the production in stage 1, $L$ different lots of common intermediate parts are made ready for the production in stage 2. They are fabricated in sequence into customized end products under the common production cycle time policy. The inventory level of common intermediate parts waiting to be fabricated in stage 2 is depicted in Figure 3.

In stage 2, after the completion of rework process ($t_{3,i}$) of each end product $i$, fixed quantity $n$ installments of the finished batch are transported to customers at a fixed interval of time in the delivery time $t_{3,i}$ (see Fig. 1). The inventory level of end products at the buyers’ side during a production cycle is depicted in Figure 4 (which is similar to Fig. 3 in [9]).

The following are additional notation used in this study (where $i = 1, 2, ..., L$, represents $L$ different products in stage 2; and $i = 0$ denotes the common intermediate part in stage 1):

- $T$ – Production cycle length, one of the decision variables,
- $n$ – Number of fixed quantity installments of the finished batch to be delivered in each cycle, the other decision variable,
- $\alpha$ – Completion rate of common intermediate part as compared to the finished product,
- $Q_i$ – Production lot size for product $i$,
- $K_i$ – Production setup cost for product $i$ in a production cycle,
- $C_i$ – Unit production cost for product $i$,
- $h_{1,i}$ – Unit holding cost for product $i$,
- $h_{2,i}$ – Holding cost per reworked item for product $i$,

Fig. 4 Inventory level of customized final products at the buyers’ side during a production cycle [9]
A two-stage EPQ-based production plan considering the postponement is proposed to satisfy annual demand from Figs. 1 to 4 (where

In stage 2, for fabrication of $L$ different customized products. From Figure 1, we observe the production cycle time as

$$T = t_{1,i} + t_{2,i} + t_{3,i} = \frac{Q_i}{\lambda_i} \quad \text{for } i = 0, 1, 2, \ldots, L$$

In stage 1, the production lot-size of common intermediate parts $Q_0$ depends on the sum of production lot sizes $Q_i$ of $L$ different products to be made in the stage 2. Therefore, we obtain the following equations (refer to Fig. 1):

$$Q_i = \lambda_i T \quad \text{for } i = 1, 2, \ldots, L$$

$$Q_0 = \sum_{i=1}^{L} Q_i = \lambda_0 T$$

$$t_{1,0} = \frac{Q_0}{P_{1,0}} = \frac{H_{1,0}}{P_{1,0} - d_{1,0}}$$

$$H_{1,0} = t_{1,0} (P_{1,0} - d_{1,0})$$

$$H_{2,0} = H_{1,0} + P_{2,0} t_{2,0} = \sum_{i=1}^{L} Q_i$$

$$t_{2,0} = \frac{x_0 Q_0}{P_{2,0}} = \frac{d_{1,0} + t_{1,0}}{P_{2,0}} = \frac{H_{2,0} - H_{1,0}}{P_{2,0}}$$

$$H_i = H_{i-1} - Q_i \quad \text{for } i = 2, 3, \ldots, L$$

$$H_L = H_{(L-1)} - Q_L = 0$$

In stage 2, for fabrication of $L$ different products we obtain the following equations directly from Figs. 1 to 4 (where $i = 1, 2, \ldots, L$):

$$t_{1,i} = \frac{Q_i}{P_{1,i}} = \frac{H_{1,i}}{P_{1,i} - d_{1,i}}$$

$$H_{1,i} = (P_{1,i} - d_{1,i}) t_{1,i}$$

$$H_{2,i} = H_{1,i} + P_{2,i} t_{2,i}$$

$$t_{2,i} = \frac{x_i Q_i}{P_{2,i}} = \frac{d_{1,i} + t_{1,i}}{P_{2,i}} = \frac{H_{2,i} - H_{1,i}}{P_{2,i}}$$

2.1 Modeling and analysis

A two-stage EPQ-based production plan considering the postponement is proposed to satisfy annual demand $\lambda_i$ of $L$ different customized products. From Figure 1, we observe the production cycle time as

$$T = t_{1,i} + t_{2,i} + t_{3,i} = \frac{Q_i}{\lambda_i} \quad \text{for } i = 0, 1, 2, \ldots, L$$

In stage 1, the production lot-size of common intermediate parts $Q_0$ depends on the sum of production lot sizes $Q_i$ of $L$ different products to be made in the stage 2. Therefore, we obtain the following equations (refer to Fig. 1):

$$Q_i = \lambda_i T \quad \text{for } i = 1, 2, \ldots, L$$

$$Q_0 = \sum_{i=1}^{L} Q_i = \lambda_0 T$$

$$t_{1,0} = \frac{Q_0}{P_{1,0}} = \frac{H_{1,0}}{P_{1,0} - d_{1,0}}$$

$$H_{1,0} = t_{1,0} (P_{1,0} - d_{1,0})$$

$$H_{2,0} = H_{1,0} + P_{2,0} t_{2,0} = \sum_{i=1}^{L} Q_i$$

$$t_{2,0} = \frac{x_0 Q_0}{P_{2,0}} = \frac{d_{1,0} + t_{1,0}}{P_{2,0}} = \frac{H_{2,0} - H_{1,0}}{P_{2,0}}$$

$$H_i = H_{i-1} - Q_i \quad \text{for } i = 2, 3, \ldots, L$$

$$H_L = H_{(L-1)} - Q_L = 0$$

In stage 2, for fabrication of $L$ different products we obtain the following equations directly from Figs. 1 to 4 (where $i = 1, 2, \ldots, L$):

$$t_{1,i} = \frac{Q_i}{P_{1,i}} = \frac{H_{1,i}}{P_{1,i} - d_{1,i}}$$

$$H_{1,i} = (P_{1,i} - d_{1,i}) t_{1,i}$$

$$H_{2,i} = H_{1,i} + P_{2,i} t_{2,i}$$

$$t_{2,i} = \frac{x_i Q_i}{P_{2,i}} = \frac{d_{1,i} + t_{1,i}}{P_{2,i}} = \frac{H_{2,i} - H_{1,i}}{P_{2,i}}$$
2.2 Cost analysis

Inventory holding costs for common intermediate parts (including perfect and imperfect quality items) during \( t_{1,0}, t_{2,0}, \) and \( t_{3,0} \), are (see Figs. 1 and 2)

\[
h_{1,0} \left[ \frac{H_{1,0} t_{1,0}^2}{2} + \left( H_{2,0} + H_{1,0} \right) \frac{t_{2,0}^2}{2} + \sum_{i=1}^{r} H_i \left( t_{i,1} + t_{2,1} \right) \right] + h_{1,0} \left[ \frac{d_{1,0} t_{1,0} t_{1,0} t_{2,0}^2}{2} \right]
\]

In stage 2, inventory holding cost for common intermediate parts waiting to be fabricated into customized end products (see Fig. 3) is

\[
\sum_{i=1}^{r} \left\{ h_{1,i} \left[ \frac{Q_i}{2} \left( t_{1,1} \right) \right] \right\}
\]

Inventory holding costs for imperfect quality items waiting to be reworked in both stages are

\[
h_{2,0} \left[ \frac{d_{1,0} t_{1,0}}{2} \left( t_{2,0} \right) \right] + \sum_{i=1}^{r} \left[ h_{2,i} \left( \frac{P_{2,i} t_{2,i}}{2} \right) \left( t_{2,i} \right) \right]
\]

In stage 2, fixed and variable delivery costs and inventory holding cost for finished product \( i \) waiting to be distributed in \( t_{3,1} \) are

\[
\sum_{i=1}^{r} \left\{ nK_{1,i} + C_{T,i} Q_i \right\} + \sum_{i=1}^{r} \left\{ h_{1,i} \left( \frac{n-1}{2n} \right) H_{2,i} t_{3,1} \right\}
\]

The stock holding cost for end product \( i \) stored at customers’ sides (see Fig. 4) is

\[
\sum_{i=1}^{r} \left\{ h_{3,i} \left[ \frac{n(D_i t_{i} + n_i)}{2} + \frac{n(n+1)}{2} I_i t_{n_i} + \frac{nI_i (t_{i,1} + t_{2,1})}{2} \right] \right\}
\]

The overall cost per cycle \( TC(T, n) \) for the proposed system, includes production setup cost, variable production cost, reworking cost, holding cost, and safety stock cost in both stages; and fixed and variable delivery costs and holding costs for stocks stored at customers’ side in stage 2. Hence, \( TC(T, n) \) is

\[
TC(T, n) = \left\{ \frac{K_0 + C_0 Q_0 + C_{R,0} x_0 Q_0 + h_{2,0} \left( \frac{d_{1,0} t_{1,0}}{2} \right) \left( t_{2,0} \right) + h_{4,0} (x_0 Q_0) T}{H_{1,0} \left[ \frac{H_{1,0} t_{1,0}^2}{2} + \left( H_{2,0} + H_{1,0} \right) \frac{t_{2,0}^2}{2} + \sum_{i=1}^{r} H_i \left( t_{i,1} + t_{2,1} \right) \right] + h_{1,0} \left[ \frac{d_{1,0} t_{1,0} t_{1,0} t_{2,0}^2}{2} \right]} \right\}
\]

Substituting Eqs. 1 to 18 in Eq. 24 and taking randomness of defective rate into account, and the long-run average system costs \( E[TCU(T, n)] \) can be derived as follows:

\[
E[TCU(T, n)] = \frac{E[TC(T, n)]}{T} = \frac{K_0 + C_0 A_0 + C_{R,0} A_0 E[x_0] + z_0 T}{T} + \sum_{i=1}^{r} \left\{ \frac{K_i}{T} + C_i A_i + C_{R,i} A_i E[x_i] + \frac{nK_{1,i} + C_{T,i} A_i}{T} + \frac{h_{1,i} T \beta_i^2}{2} \left( \delta_{2,i} - \delta_{1,i} \right) }{2P_{2,i}} \right\}
\]

where

\[
\sum_{i=1}^{r} \left\{ \frac{h_{2,i} T \beta_i^2 E[x_i]^2}{2P_{2,i}} + \frac{h_{4,i} T \beta_i^2}{2} \left[ \frac{1}{P_{1,i}} + \frac{E[x_i]}{r_{2,i}} + \frac{\delta_{1,i}}{n} \right] + Th_{4,i} A_i E[x_i] \right\}
\]
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\[
E_0 = \left\{ \frac{h_{1,0}^2 \lambda_0^2}{2} \left[ \frac{1}{P_{1,0}} + \frac{2E[x_0]}{P_{2,0}} - \frac{E[x_0]^2}{P_{2,0}} \right] + \frac{h_{2,0}^2 \lambda_0^2 E[x_0]^2}{2P_{2,0}} \right\};
\]
\[
\delta_{1,i} = \frac{1}{\lambda_i} \frac{1}{p_{1,i}} - \frac{E[x_i]}{p_{2,i}}, \text{ and } \delta_{2,i} = \left[ \frac{1}{\lambda_i} \frac{E[x_i]^2}{p_{2,i}} + \frac{1}{p_{1,i}} + \frac{E[x_i]}{p_{2,i}} \right] \text{ for } i = 1, 2, ..., L \tag{26}
\]

3. Convexity and the optimal decision

Upon obtaining the long-run average system costs \(E[TCU(T, n)]\), we then prove it is a convex function by applying the Hessian matrix equations [45] to verify that Eq. 27 holds.

\[
[T \ n] \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial T^2} \right) \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial T \partial n} \right) \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial n^2} \right) > 0 \tag{27}
\]

From Eq. 25 we obtain

\[
\frac{\partial E[TCU(T, n)]}{\partial T} = \left( \frac{-K_0}{T^2} + z_0 \right) + \sum_{i=1}^{L} \left\{ \frac{-K_{i,1}}{T^2} + \frac{h_{2,i}^2 \lambda_i^2 E[x_i]_i}{2p_{2,i}} + \frac{h_{3,i}^2 \lambda_i^2}{2n} \left[ \frac{1}{p_{1,i}} + \frac{E[x_i]}{p_{2,i}} + \frac{\delta_{2,i}}{n} \right] \right\} \tag{28}
\]

\[
\frac{\partial E[TCU(T, n)]}{\partial T^2} = \frac{2K_0}{T^3} + \sum_{i=1}^{L} \left\{ \frac{2K_i}{T^3} + \frac{2h_{2,i}^2 \lambda_i^2 E[x_i]_i}{2n} \right\} \tag{29}
\]

\[
\frac{\partial E[TCU(T, n)]}{\partial n} = \sum_{i=1}^{L} \left\{ \frac{h_{2,i}^2 \lambda_i^2 E[x_i]^2}{2p_{2,i}} + \frac{h_{3,i}^2 \lambda_i^2}{2n} \left[ \frac{1}{p_{1,i}} + \frac{E[x_i]}{p_{2,i}} + \frac{\delta_{2,i}}{n} \right] \right\} \tag{30}
\]

\[
\frac{\partial^2 E[TCU(T, n)]}{\partial T \partial n} = \sum_{i=1}^{L} \left\{ \frac{-K_{i,1}}{T^2} + \frac{T_{i,2} \lambda_i^2}{2n^2} \left[ (h_{1,i} - h_{3,i}) \delta_{1,i} \right] \right\} \tag{31}
\]

Substituting Eqs. 29, 31, and 32 in Eq. 27, we obtain

\[
[T \ n] \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial T^2} \right) \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial T \partial n} \right) \cdot \left( \frac{\partial^2 E[TCU(T, n)]}{\partial n^2} \right) \cdot \left( \frac{T}{n} \right) = \frac{2K_0}{T} + \sum_{i=1}^{L} \frac{2K_i}{T} > 0 \tag{33}
\]

Because \(K_0, K_i\), and \(T\) are all positive, we find Eq. 33 is positive. Hence, \(E[TCU(T, n)]\) is a strictly convex function for all \(T\) and \(n\) different from zero. In order to simultaneously determine production-shipment decision for the proposed system, we can solve the linear system of first derivatives of \(E[TCU(T, n)]\) with respect to \(T\) and \(n\), respectively, by setting these partial derivatives equal to zero. With further derivations we find

\[
T^* = \sqrt{\frac{K_0 + \sum_{i=1}^{L} (K_i + nK_{i,1})}{z_0 + \sum_{i=1}^{L} \left\{ \frac{h_{1,i}^2 \lambda_i^2}{2} \left( \delta_{2,i} - \delta_{1,i} \right), \frac{h_{2,i}^2 \lambda_i^2 E[x_i]^2}{2p_{2,i}} \right\} + \frac{h_{3,i}^2 \lambda_i^2}{2n} \left[ \frac{1}{p_{1,i}} + \frac{E[x_i]}{p_{2,i}} + \frac{\delta_{2,i}}{n} \right] + h_{4,i} \lambda_i E[x_i]}} \tag{34}
\]

and

\[
T^* = \sqrt{\frac{(K_0 + \sum_{i=1}^{L} K_{i,1}) \frac{\lambda_i^2}{2} \left( (h_{3,i} - h_{1,i}) \left( \delta_{1,i} \right) \right)}{z_0 + \sum_{i=1}^{L} \left\{ \frac{h_{1,i}^2 \lambda_i^2}{2} \left( \delta_{2,i} + \frac{h_{2,i}^2 \lambda_i^2 E[x_i]^2}{2p_{2,i}} + h_{4,i} \lambda_i E[x_i] \right) \right\} + \frac{h_{3,i}^2 \lambda_i^2}{2n} \left[ \frac{1}{p_{1,i}} + \frac{E[x_i]}{p_{2,i}} + \frac{\delta_{2,i}}{n} \right]}} \tag{35}
\]
4. Numerical example and discussion

The following numerical example is used to show the practical uses of research results obtained in the previous section. Consider a manufacturer must fabricate five different products and they share a common intermediate part that has completion rate $\alpha = 0.5$ (i.e., halfway done). To ease comparison efforts for readers, we reconsider a numerical example used in a prior study [9] regarding optimization of a single-stage multi-product system without adopting postponement in its production. Annual production rates of five end products $P_{1,j} = 58,000$, $59,000$, $60,000$, $61,000$, and $62,000$ units, respectively; annual demands $\lambda_i = 3,000$, $3,200$, $3,400$, $3,600$, and $3,800$ units, respectively; annual reworking rates $P_{2,j} = 46,400$, $47,200$, $48,000$, $48,800$, and $49,600$ units, respectively; setup costs $K_i = $17,000, $17,500$, $18,000$, $18,500$, and $19,000$, respectively; unit fabrication costs $C_i = $80, $90$, $100$, $110$, and $120$, respectively; the defective rates $x_i$ follow uniform distribution over the intervals $[0, 0.05]$, $[0, 0.10]$, $[0, 0.15]$, $[0, 0.20]$, and $[0, 0.25]$, respectively; and unit reworking costs $C_{R,i} = $50, $55$, $60$, $65$, and $70$, respectively. Based on common intermediate part’s completion rate $\alpha = 0.5$, a straightforward relationship $1/\alpha$ is assumed for its relevant production rates. Hence, in the proposed two-stage single-machine production scheme we have $P_{1,0} = (1/\alpha)^*(the\ mean\ of\ P_{1,i}'s) = 120,000$ and $P_{2,0} = (1/\alpha)^*(the\ mean\ of\ P_{2,i}'s) = 96,000$.

The relationship between common intermediate part’s relevant costs and its completion rate $\alpha$ can either be linear or nonlinear. Both cases are investigated in the following subsections.

4.1 Case 1: Analysis of linear relationship of cost relevant variables

If the relationship between practical fabrication related cost of common intermediate part (or called ‘the value’ of common part) and its completion rate $\alpha$ is linear, then for $\alpha = 0.5$ we have the following linear-based relevant values of variables in our proposed system:

- $C_0$ – $40$, unit fabrication cost for common intermediate part,
- $K_0$ – $8,500$, setup cost for common intermediate part,
- $C_{R,0}$ – $25$, unit reworking cost for common intermediate part,
- $h_{1,0}$ – $5$, unit holding cost for common intermediate part,
- $h_{4,0}$ – $5$, unit safety stock cost for common intermediate part,
- $h_{2,0}$ – $15$, unit holding cost for common intermediate part during the reworking processes,
- $K_i$ – Setup costs of end products are $8,500$, $9,000$, $9,500$, $10,000$, and $10,500$ respectively,
- $x_0$ – $[0, 0.04]$, the interval uniformly distributed defective rate in the production of common intermediate part,
- $C_i$ – Unit production costs of end products are $40$, $50$, $60$, $70$, and $80$, respectively,
- $h_{1,i}$ – Unit holding costs of end products are $10$, $15$, $20$, $25$, and $30$, respectively,
- $P_{1,i}$ – Annual production rates of five end products are $112,258$, $116,066$, $120,000$, $124,068$, and $128,276$ units, respectively; they are simply calculated by $P_{1,i} = 1/(1/\alpha - 1/P_{1,0})$,
- $x_i$ – End items’ defective rates follow the uniform distribution over the intervals $[0, 0.06]$, $[0, 0.11]$, $[0, 0.16]$, and $[0, 0.21]$, respectively,
- $C_{R,i}$ – Unit reworking costs of end products are $25$, $30$, $35$, $40$, and $45$, respectively,
- $P_{2,i}$ – Annual reworking rates of five end products are $89,806$, $92,852$, $96,000$, $99,254$, and $102,621$ units, respectively; they are simply calculated by $P_{2,i} = 1/(1/\alpha - 1/P_{2,0})$,
- $h_{2,i}$ – Unit holding cost per reworked items of end products are $30$, $35$, $40$, $45$, and $50$, respectively,
- $K_{1,i}$ – Fixed delivery costs per shipment: $1,800$, $1,900$, $2,000$, $2,100$, and $2,200$, respectively,
- $C_{T,j}$ – Unit delivery costs of end items are $0.1$, $0.2$, $0.3$, $0.4$, and $0.5$, respectively,
- $h_{3,i}$ – Unit holding costs at the customer’s side are $70$, $75$, $80$, $85$, and $90$, respectively,
- $h_{4,0}$ – Unit safety stock costs of end products are $10$, $15$, $20$, $25$, and $30$, respectively.
First, the annual demand for common intermediate parts $\lambda_0 = 17,000$ can be obtained by applying Eqs. 2 and 3. Then, by calculating Eqs. 34, 35, and 25, we derive the optimal number of deliveries $n^* = 3$, optimal production cycle time $T^* = 0.4614$ (years), and the expected system costs per unit time $E[TCU(T^*, n^*)] = 2,145,834$. Figure 5 depicts the effects of variations of the production cycle time $T$ on the expected system costs $E[TCU(T, n)]$.

The behavior of $E[TCU(T, n)]$ with respect to the common intermediate part’s completion rate $\alpha$ is exhibited in Figure 6. It can be seen that as the completion rate $\alpha$ increases, the long-run expected system costs $E[TCU(T, n)]$ decreases, and the proposed model realizes a system cost savings of 3.76% at $\alpha = 0.5$ (i.e., system costs decreased from $2,229,658$ [9] to $2,145,834$) as compared to that in prior study which used a single-stage production scheme. This analytical result demonstrates that the proposed two-stage multi-item production scheme with delayed differentiation is a considerably beneficial model for manufacturers who must meet demands for multiple products that share a common intermediate part.

Figure 7 shows the effects of variations of common part’s completion rate $\alpha$ on the optimal production cycle time $T^*$. As the completion rate $\alpha$ increases, the optimal cycle time $T^*$ decreases significantly, and in the proposed model optimal cycle time $T^*$ is reduced by 25.5% at $\alpha = 0.5$ (i.e., it decreases from 0.6193 [9] to 0.4614 (years)) as compared to that in prior study which used a single-stage production scheme. Such an analytical result indicates our proposed two-stage multi-item production scheme with delayed differentiation provides a shorter cycle time (or faster response time) than that in a conventional one-stage multi-item system [9].

![Fig. 5](image5.png)  
**Fig. 5** The effects of variations of the production cycle time $T$ on the expected system costs $E[TCU(T, n)]$.

![Fig. 6](image6.png)  
**Fig. 6** The behavior of $E[TCU(T, n)]$ with respect to the common intermediate part’s completion rate $\alpha$.

![Fig. 7](image7.png)  
**Fig. 7** The effects of variations of common part’s completion rate $\alpha$ on the optimal production cycle time $T^*$.

### 4.2 Case 2: Analysis of nonlinear relationship of cost relevant variables

In this section, we demonstrate that the proposed model is capable of analyzing any given nonlinear relationship between the common part’s relevant costs and its completion rate $\alpha$. For instance, if a nonlinear relationship of ‘$\alpha^{(1/3)}$’ between common part’s relevant costs and $\alpha$ is known, then $C_0 = [\alpha^{(1/3)}]C_1 = [(0.5)^{(1/3)}]$ $\approx 0.80 = $63, so it obviously has higher production...
cost (or called value) than that in the linear relationship case (which is $40). Apply the similar computation we have the following values of other relevant parameters: \( C_{R0} = 40, K_0 = 13,493, h_{1,0} = h_{4,0} = 8, \) and \( h_{2,0} = 24. \) Assume the following parameters’ values remain the same as stated in subsection 4.1: \( P_{1,0} = 120,000, P_{2,0} = 96,000, \) and \( x_0 = [0, 0.04]. \) Accordingly, in stage 2 we obtain the values of other variables as follows: \( C_i = $17, \) \( $27, \) \( $37, \) \( $47, \) and \( $57, \) respectively; \( K_i = $3,507, $4,007, $4,507, $5,007, \) and \( $5,507, \) respectively; \( C_{Ri} = $10, $15, $20, $25, \) and \( $30, \) respectively; and \( x_i \) follows the uniform distribution over the intervals \([0, 0.01], [0, 0.06], [0, 0.11], [0, 0.16], \) and \([0, 0.21], \) respectively.

We apply Eqs. 34, 35, and 25 to obtain the optimal number of shipments \( n^* = 3, \) the optimal production cycle time \( T^* = 0.4005 \) (years), and the expected system costs \( E[TCU(T^*, n^*)] = $2,093,253. \) Figure 8 depicts the behavior of \( E[TCU(T,n)] \) with respect to the common part’s completion rate \( \alpha \) under both linear and nonlinear relationships. In nonlinear relationship case, as the common part’s completion rate \( \alpha \) increases, the expected system costs \( E[TCU(T,n)] \) decreases, and it indicates that \( E[TCU(T,n)] \) is decreased by 2.45 % at \( \alpha = 0.5 \) (i.e., system costs declined from $2,145,834 to $2,093,253) compared to that in the earlier linear case. The analytical results demonstrate that the proposed two-stage multi-item production scheme with delayed differentiation is a greatly beneficial model to manufacturers who have to meet demands for multiple products that share a common intermediate part.

Figure 9 illustrates the behavior of the optimal production cycle time \( T^* \) with respect to the common part’s completion rate \( \alpha \) under both linear and nonlinear relationships. As completion rate \( \alpha \) increases, the optimal production cycle time \( T^* \) decreases significantly, and in the nonlinear case, the optimal cycle time \( T^* \) is shortened by 13.20 % at \( \alpha = 0.5 \) (i.e., it reduces from 0.4614 to 0.4005) compared to that in the earlier linear case. Therefore, it demonstrates that the proposed two-stage multi-item production scheme with delayed differentiation is a considerably beneficial model (in terms of faster response cycle time) for manufacturers who have to meet demands for multiple products that share a common intermediate part.

Furthermore, the analytical results reveals that if the common part’s relevant costs are higher (e.g., having a nonlinear relationship \( \alpha^2(1/3) \) rather than the linear one), then the optimal cycle time \( T^* \) reduces significantly compared to that in the linear case.

5. Conclusion

Inspired by the potential benefits derived from applying delayed differentiation to multi-product systems, and with the aim of providing managers of transnational enterprises with information to assist them in achieving the key operational goals such as maximizing machine utilization, ensuring product quality, lowering overall operating costs, and shortening response time, this
study explores the effect of delayed differentiation on a multi-product vendor-buyer integrated inventory system with rework, using a single machine production scheme.

Using mathematical modeling and optimization methods, we derive the closed-form optimal replenishment cycle time and delivery decisions and demonstrate the practical use of our results through a numerical example. The results reveal that our proposed multi-product fabrication scheme with delayed differentiation strategy is considerably beneficial in saving expected system costs and reducing replenishment cycle time. Further analysis also indicates that when the common intermediate part’s value is higher (e.g., having a nonlinear relationship $\alpha^\gamma(1/3)$ rather than the linear one), both the expected system costs and production cycle time reduces significantly compared to that in the linear case. For future study, to explore and compare the effects of the dual-machine production scheme on the optimal operating policies of the same system would be an interesting direction.

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