APEM journal

Advances in Production Engineering & Management
Volume 14 | Number 1 | March 2019 | pp 112–124
https://doi.org/10.14743/apem2019.1.315

ISSN 1854-6250
Journal home: apem-journal.org
Original scientific paper

A multi-product pricing and inventory model with production rate proportional to power demand rate

Keshavarzfard, R.[a], Makui, A.[b], Tavakkoli-Moghaddam, R.[c,d,*]

[a]School of Industrial Engineering, South Tehran Branch, Islamic Azad University, Tehran, Iran
[b]School of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran
[c]School of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran
[d]Arts et Métiers ParisTech, LCFC, Metz, France

ABSTRACT

This paper deals with an economic lot size model when demand follows a power law and changes with time, owing to the fact that this kind of pattern is suitable for so many real situations. Production rate is presumed to be proportional to demand rate. Also since that demand is price sensitive in reality, we suppose that demand decreases linearly with price. With regard to these points, in this article a joint pricing and inventory model is presented where demand depends on time with a power pattern and price linearly, production rate changes pro rata with demand rate and multiple items are considered. The principal consideration of the study is to satisfy the demand and optimize the profit for all items in the system, simultaneously. Setup, holding, backlogging and production costs are involved in the inventory system. The aim is to maximize total profit function and achieve optimum values of scheduling period, reorder point and price. Employing mathematical modelling and optimization methods, the existence of the optimal solutions is proved, and then a simple heuristic algorithm is presented to maximize total inventory profit and determine the best values of variables. A numerical analysis is carried out to illustrate the applications of the proposed models.

ARTICLE INFO

Keywords:
Pricing model;
Inventory model;
Economic production quantity (EPQ);
Backordered shortages;
Power demand pattern

**Corresponding author:*
tavakoli@ut.ac.ir
(Tavakkoli-Moghaddam, R.)

Article history:
Received 14 August 2017
Revised 15 February 2019
Accepted 24 February 2019

1. Introduction

The constant demand rate is one of the presumptions of the conventional Economic Production Quantity (EPQ) model proposed by Taft [1] that is not common in practice. In the most real situations demand of consumers varies with time. Therefore, many researchers have worked on time-varying inventory models. Barbosa and Friedman [2] worked on a special type of time-varying demand to find a complete solution. Mitra *et al*. [3] used a demand pattern that increases or decreases with time in an inventory system. A multi-period inventory model is presented by Sakaguchi [4] assuming that demand varies with time. Also Khanra *et al*. [5] proposed a quadratic-time demand for an EOQ with shortages and allowable delay in payments. Sarkar *et al*. [6] studied a production system with demand depending on time and price together with considering the influence of inflation and reliability. Escuín *et al*. [7] discussed inventory models using stochastic and time varying demand for a paper manufacturer.

In this paper, we use a kind of demand pattern called power demand. It can be applied for the situations that a high percentage of requests for products happen at the beginning of the cycle, like breads, yogurt, fruits, prepared food, vegetables, fresh meat, etc., or at the end of the cycle, like tea, coffee, sugar, oil, etc. Naddor [8] introduced an inventory system with a power demand pattern. Lee and Wu [9] presented an optimal order quantity model allowing for shortages, power demand and deteriorating items. Singh *et al.* [10] studied an EOQ model where demand has a power pattern and partial backlogging and perishable products are allowed. Sicilia *et al.* [11] analyzed systems with deterministic power demand pattern and different situations: with and without shortages, with full backordering or completely lost sales. Rajeswari and Vanjikkodi [12] worked on an inventory system considering Weibull deterioration and demand following a power law. Mishra and Singh [13] proposed an economic ordering model with demand having a power pattern, perishable units and partially backordered shortages. Sicilia *et al.* [14] offered an optimal order size model taking into account power demand along with constant deterioration rate. Sicilia *et al.* [15] presented an optimum lot size model in which demand has a power pattern, production rate changes pro rata with demand rate and backordered items are allowed. Sicilia *et al.* [16] studied lot size models where demand follows a power law and the replenishment rate is uniform. San-José *et al.* [17] analyzed an inventory system with demand following a power law and partial backordering.

Demand of consumers is price sensitive, so that pricing is one of the most important decisions in a company. Tripathi *et al.* [18] investigated inventory dependent demand with a power rate and holding cost functions for two situations. Lau *et al.* [19] worked on a joint ordering and pricing problem with deteriorating products and a two period life. Yang *et al.* [20] studied a model considering price dependent demand, deterioration and partial backordering. Banerjee and Sharma [21] investigated an inventory model in which seasonal demand varies with time and price. Soon [22] developed a review of multi-products pricing models. Maihami and Kamalabadi [23] proposed a joint inventory and pricing system with deteriorating products, price dependent demand and partial backordering. Shavandi *et al.* [24] presented a constrained inventory and pricing model with multiple items. Pal *et al.* [25] studied an economic ordering model with multiple products and price break sensitive demand. Zhu [26] worked on joint production, replenishment and pricing policies where demand is random and price sensitive and there is a supply disruption. Qin *et al.* [27] employed joint lot-sizing and pricing policies for fresh products with deterioration. Liu *et al.* [28] studied a joint investment and pricing problem for perishable products considering price and quality dependent demand. Panda *et al.* [29] presented a replenishment and pricing problem for a high tech item in a dual-channel supply chain. Alfares and Ghaithan [30] worked on a pricing and inventory system considering price dependent demand and time-varying holding cost along with quantity discounts. Chiu *et al.* [31] investigated the impact of delayed differentiation on a vendor-buyer system with rework and multiple items. Gholamian and Heydari [32] developed a mixed integer stochastic programming model by using METRIC stochastic relations in a location-routing-inventory problem.

The remainder of this paper is organized as follows. In Section 2 problem definition and mathematical model are presented. Concavity and optimal decision are proposed in section 3. A procedure for determining the optimum values can be found in section 4. Numerical analysis is presented in section 5. Finally, in section 6 some conclusions and future researches are given.

2. Problem definition and mathematical model

Consider a factory which produces N different items (where $i = 1,2,..,N$). Each item has an average demand ν_i that must be satisfied. The demand varies with time with a power pattern and decreases linearly with price. Also the production rate changes pro rata with demand rate. The manager desires to satisfy the customer demand and optimize the total profit of the system, simultaneously. The inventory cycle starts with s units of net stock at time 0. At the same time $t = 0$, production begins with the production rate $P_i(t)$, at time t_{1i} reaches zero and continues until $t = \tau_i$ for each product i, consequently the replenishment quantity Q_i will be produced. Also during the interval $[0, \tau_i]$, the inventory level of product i increases at a rate $P_i(t) - D_i(t)$. After that the stock level decreases up to $t = t_{2i}$ according to demand. Finally, during the interval $[t_{2i}, T]$, demand is backlogged. Assume that $I_{1i}(t)$ and $I_{2i}(t)$ are the on-hand inventory levels of item i at time t in the intervals $[0, \tau_i]$ and $[\tau_i, T]$, respectively. The scheduling period, the backorder size and the selling price are three decision variables of the system. In the following, an approach is presented to find the optimal values.

The following notation will be applied in the rest of the paper.

T	Scheduling period (time)
N	Number of items
Q_i	Production quantity for item i (units)
s_i	Reorder point for item i (units)
D_i	Demand quantity during the inventory cycle for item i (units)
τ_i	Production period length for item i (time)
ν_i	Average demand $\nu_i = D_i/T$ for item i (units)
Λ_i	Production setup cost for item i ($/replenishment)
p_i	Selling price for item i, where $p_i > c_i$ ($/unit)
c_i	Unit producing cost for item i ($/unit)
h_i	Unit carrying cost for item i ($/unit/unit time)
w_i	Unit backlogging cost for item i ($/unit/unit time)
$I_{1i}(t)$	Net stock level at time t for item i $(0 \le t \le \tau_i)$
$I_{2i}(t)$	Net stock level at time t for item i $(0 \le t \le T)$
$P_i(t)$	Production rate at time t for item i $(0 \le t \le \tau_i)$
$D_i(t)$	Demand rate at time t for item i $(0 \le t \le T)$
CH_i	Cost of holding for item i ($/unit time)
CB_i	Cost of backordering for item i ($/unit time)
CS_i	Cost of setup for item i ($/unit time)
CP_i	Cost of production for item i ($/unit time)
SR_i	Sales revenue for item i ($/unit time)
$TP_i(p_i, s_i, T)$	Total profit for item i ($/unit time)
$\Pi(\vec{p}, \vec{s}, T)$	Total profit of the system ($/unit time)

The following assumptions are considered:

- The planning horizon is infinite.
- Shortages are fully backordered.
- Demand rate changes with time with a power pattern and decreases linearly with price. So, the demand rate is assumed to be $D_i(t) = (a_i - b_i p_i)\frac{\nu_i}{\kappa_i}(\frac{t}{T})^{1/\kappa_i - 1}$, with $0 \le t \le T$, where $0 < \kappa_i < \infty$, $a_i > 0$, $b_i > 0$.
- Multiple items are assumed for the inventory system.
- The demand rate is less than the production rate for each item.
- The production rate $P_i(t)$ is proportional to demand rate $D_i(t)$ for each item i at any time $t(0 \le t \le \tau_i)$ and is defined by $P_i(t) = \alpha_i D_i(t)$ with $\alpha_i > 1$.

The demand over the scheduling period $[0, T]$ is defined by the following:

$$\int_0^T D_i(t)dt = \int_0^T (a_i - b_i p_i)\frac{\nu_i}{\kappa_i}\left(\frac{t}{T}\right)^{1/\kappa_i - 1} dt = (a_i - b_i p_i)\nu_i T \tag{1}$$

The differential equations governing the system are as following:

$$\frac{dI_{1i}(t)}{dt} = P_i(t) - D_i(t) = (\alpha_i - 1)D_i(t) = (\alpha_i - 1)(a_i - b_i p_i)\frac{\nu_i}{\kappa_i}\left(\frac{t}{T}\right)^{1/\kappa_i - 1}, \tag{2}$$

$$I_{1i}(0) = s_i, \quad 0 \le t \le \tau_i$$

$$\frac{dI_{2i}(t)}{dt} = -D_i(t) = -(a_i - b_i p_i)\frac{\nu_i}{\kappa_i}\left(\frac{t}{T}\right)^{1/\kappa_i - 1}, \quad I_{2i}(T) = s_i, \quad \tau_i \le t \le T \tag{3}$$

With regard to the boundary conditions, $I_{1i}(0) = I_{2i}(T) = s_i$, Eq. 1 and Eq. 2 are solved and the solutions are:

$$I_{1i}(t) = s_i + (\alpha_i - 1)(a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}, \quad 0 \le t \le \tau_i \tag{4}$$

$$I_{2i}(t) = s_i + (a_i - b_i \wp_i)\nu_i T - (a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}, \quad \tau_i \le t \le T \tag{5}$$

At time τ_i the production cycle of lot size is finished and the maximum inventory level can be calculated by both Eq. 4 and Eq. 5. So that τ_i and stock level at τ_i are respectively:

$$\tau_i = \frac{T}{{\alpha_i}^{\kappa_i}} \tag{6}$$

$$I_i(\tau_i) = s_i + \frac{\alpha_i - 1}{\alpha_i}(a_i - b_i \wp_i)\nu_i T \tag{7}$$

Also the lot size Q_i for product i is calculated by

$$Q_i = \int_0^{\tau_i} P_i(t)dt = \int_0^{\tau_i} \alpha_i (a_i - b_i \wp_i) \frac{\nu_i}{\kappa_i} \left(\frac{t}{T}\right)^{1/\kappa_i - 1} dt = (a_i - b_i \wp_i)\nu_i T \tag{8}$$

As it was expected, the lot quantity is equal to the demand of scheduling period. We assume that $I(\tau_i) \ge 0$ and $s_i \le 0$, So that $-(a_i - b_i \wp_i)\frac{(\alpha_i - 1)}{\alpha_i} rT \le s_i \le 0$. Suppose that the stock level reaches zero in the production period at time t_{1i}. Since $I_{1i}(t_{1i}) = 0$, from Eq. 4 we obtain t_{1i} for item i according to decision variables s_i and T:

$$t_{1i} = \left(\frac{-s_i}{(\alpha_i - 1)(a_i - b_i \wp_i)\nu_i T}\right)^n T \tag{9}$$

The net stock level of interval $[\tau_i, T]$ reaches zero at time t_{2i}. Solving equation $I_{2i}(t_{2i}) = 0$, t_{2i} can be obtained for item i according to decision variables s_i and T:

$$t_{2i} = \left(1 + \frac{s_i}{(a_i - b_i \wp_i)\nu_i T}\right)^n T \tag{10}$$

We consider four various cost in the inventory system for each product i as follows. Note that the average number of production runs is $\frac{1}{T}$.

The carrying cost:

$$\begin{aligned} CH_i &= \frac{h_i}{T}\Big(\int_{t_{1i}}^{\tau_i} \left[s_i + (\alpha_i - 1)(a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}\right] dt \\ &+ \int_{\tau_i}^{t_{2i}} \left[s_i + (a_i - b_i \wp_i)\nu_i T - (a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}\right] dt\Big) = \\ &= \Bigg(\frac{(s_i + (a_i - b_i \wp_i)\nu_i T)^{\kappa_i + 1}}{(\kappa_i + 1)(a_i - b_i \wp_i)^{\kappa_i} {\nu_i}^{\kappa_i} T^{\kappa_i}} + \frac{(-s_i)^{\kappa_i + 1}}{(\kappa_i + 1)(\alpha_i - 1)^{\kappa_i}(a_i - b_i \wp_i)^{\kappa_i} {\nu_i}^{\kappa_i} T^{\kappa_i}} \\ &- \frac{(a_i - b_i \wp_i)\nu_i T}{(\kappa_i + 1){\alpha_i}^{\kappa_i}}\Bigg) h_i \end{aligned} \tag{11}$$

The shortage cost:

$$\begin{aligned} CB_i &= -\frac{w_i}{T}\Bigg(\int_0^{t_{1i}} \left[s_i + (\alpha_i - 1)(a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}\right] dt \\ &+ \int_{t_{2i}}^{T} \left[s_i + (a_i - b_i \wp_i)\nu_i T - (a_i - b_i \wp_i)\nu_i T \left(\frac{t}{T}\right)^{1/\kappa_i}\right] dt\Bigg) = \\ &= \Bigg(\frac{(s_i + (a_i - b_i \wp_i)\nu_i T)^{\kappa_i + 1}}{(\kappa_i + 1)(a_i - b_i \wp_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} + \frac{(-s_i)^{\kappa_i + 1}}{(\kappa_i + 1)(\alpha_i - 1)^{\kappa_i}(a_i - b_i \wp_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} - \frac{(a_i - b_i \wp_i)\nu_i T}{(\kappa_i + 1)} \\ &- s_i\Bigg) w_i \end{aligned} \tag{12}$$

The production cost:

$$CP_i = c_i \frac{Q_i}{T} = (a_i - b_i p_i) c_i \nu_i \tag{13}$$

The setup cost:

$$CS_i = \frac{\Lambda_i}{T} \tag{14}$$

The sales revenue is:

$$SR_i = p_i \frac{Q_i}{T} = (a_i - b_i p_i) p_i \nu_i \tag{15}$$

The total profit of product i can be calculated then as:

$$\begin{aligned} TP_i(p_i, s_i, T) &= SR_i - CH_i - CB_i - CP_i - CS_i = \\ &= (a_i - b_i p_i) p_i \nu_i - \frac{(s_i + (a_i - b_i p_i)\nu_i T)^{\kappa_i+1}(h_i + w_i)}{(\kappa_i + 1)(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} \\ &\quad - \frac{(-s_i)^{\kappa_i+1}(h_i + w_i)}{(\kappa_i + 1)(\alpha_i - 1)^{\kappa_i}(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} + \frac{(a_i - b_i p_i)\nu_i T}{(\kappa_i + 1)} \left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i \right) \\ &\quad + w_i s_i - \frac{\Lambda_i}{T} - (a_i - b_i p_i) c_i \nu_i \end{aligned} \tag{16}$$

Now we define $\vec{p} = (p_1, p_2, \dots, p_N)$ and $\vec{s} = (s_1, s_2, \dots, s_N)$. So that the total profit of the system can be obtained as follows:

$$\Pi(\vec{p}, \vec{s}, T) = \sum_{i=1}^{N} TP_i(p_i, s_i, T) \tag{17}$$

3. Concavity and the optimal decision

With regard to the purpose of this paper that is finding the best production policies to maximize the total profit per unit time for the multi-product inventory system, we first prove that for any given $\vec{p}$, the optimum solution of $(\vec{s}, T)$ not only exists but also is unique. Because $\Pi(\vec{p}, \vec{s}, T)$is a function of $\vec{p}, \vec{s}$ and T, so for any given $\vec{p}$, the essential conditions to maximize the total profit per unit time is equaling partial derivatives of the $\Pi(\vec{p}, \vec{s}, T)$ to zero, with respect to decision variables s_i and T, simultaneously. Thus,

$$\frac{\partial \Pi(\vec{p}, \vec{s}, T)}{\partial s_i} = \frac{(-s_i)^{\kappa_i}(h_i + w_i)}{(\alpha_i - 1)^{\kappa_i}(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} - \frac{(s_i + (a_i - b_i p_i)\nu_i T)^{\kappa_i}(h_i + w_i)}{(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i}} + w_i = 0 \quad \forall i \tag{18}$$

$$\begin{aligned} \frac{\partial \Pi(\vec{p}, \vec{s}, T)}{\partial T} &= \sum_{i=1}^{N} \frac{\kappa_i (-s_i)^{\kappa_i+1}(h_i + w_i)}{(\kappa_i + 1)(\alpha_i - 1)^{\kappa_i}(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i+1}} \\ &- \sum_{i=1}^{N} \frac{(s_i + (a_i - b_i p_i)\nu_i T)^{\kappa_i}((a_i - b_i p_i)\nu_i T - \kappa_i s_i)(h_i + w_i)}{(\kappa_i + 1)(a_i - b_i p_i)^{\kappa_i} \nu_i^{\kappa_i} T^{\kappa_i+1}} \\ &+ \sum_{i=1}^{N} \frac{(a_i - b_i p_i)\nu_i}{(\kappa_i + 1)} \left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i \right) + \frac{\sum_{i=1}^{N} \Lambda_i}{T^2} = 0 \end{aligned} \tag{19}$$

Theorem 1: For any given $\vec{p}$, The total profit function $\Pi(\vec{p}, \vec{s}, T)$ is concave.

Proof: Please see Appendix A.

Defining a new variable x_i by $x_i = \frac{-s_i}{(a_i - b_i p_i)\nu_i T}$, the region $-(a_i - b_i p_i)\frac{(\alpha_i - 1)}{\alpha_i} rT \le s_i \le 0$ is equivalent to $0 \le x_i \le \frac{\alpha_i - 1}{\alpha_i}$ and Eq. 18 and Eq. 19 are respectively equivalent to

$$(1 - x_i)^{\kappa_i} - \frac{x_i^{\kappa_i}}{(\alpha_i - 1)^{\kappa_i}} - \frac{w_i}{h_i + w_i} = 0 \tag{20}$$

$$\sum_{i=1}^{N} \frac{\kappa_i x_i^{\kappa_i + 1}(a_i - b_i p_i)\nu_i(h_i + w_i)}{(\kappa_i + 1)(\alpha_i - 1)^{\kappa_i}} - \sum_{i=1}^{N} \frac{(1 - x_i)^{\kappa_i}(1 + \kappa_i x_i)(a_i - b_i p_i)\nu_i(h_i + w_i)}{(\kappa_i + 1)} + \sum_{i=1}^{N} \frac{(a_i - b_i p_i)\nu_i}{(\kappa_i + 1)}\left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i\right) + \frac{\sum_{i=1}^{N} \Lambda_i}{T^2} = 0 \tag{21}$$

Proposition: There is a unique solution x_i^* for the function $(1 - x_i)^{\kappa_i} - \frac{x_i^{\kappa_i}}{(\alpha_i - 1)^{\kappa_i}} - \frac{w_i}{h_i + w_i} = 0$, on the interval $(0, \frac{\alpha_i - 1}{\alpha_i})$.

Proof: The proof is similar to proposition 1 in [15].

The optimal solution of Eq. 20 can be obtained using any numerical method like the Newton-Raphson method (see, i.e. [33]). Also from Eq. 20 we have

$$\frac{x_i^{\kappa_i}}{(\alpha_i - 1)^{\kappa_i}} = (1 - x_i)^{\kappa_i} - \frac{w_i}{h_i + w_i} = 0 \tag{22}$$

Substituting Eq. 22 in Eq. 23 we obtain following equation

$$-\sum_{i=1}^{N} \frac{(a_i - b_i p_i)\nu_i(h_i + w_i)(1 - x_i)^{\kappa_i}}{(\kappa_i + 1)} - \sum_{i=1}^{N} \frac{\kappa_i(a_i - b_i p_i)\nu_i w_i x_i}{(\kappa_i + 1)} + \sum_{i=1}^{N} \frac{(a_i - b_i p_i)\nu_i}{(\kappa_i + 1)}\left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i\right) + \frac{\sum_{i=1}^{N} \Lambda_i}{T^2} = 0 \tag{23}$$

Then substituting the optimal solution x_i^* obtained by Eq. 20 in Eq. 23, the best cycle length T^* for a given $\vec{p}$ is

$$T^* = \sqrt{\frac{\sum_{i=1}^{N} \Lambda_i}{\sum_{i=1}^{N}\left[\left(\frac{(a_i - b_i p_i)\nu_i}{(\kappa_i + 1)}\right)\left((h_i + w_i)(1 - x_i^*)^{\kappa_i} + \kappa_i w_i x_i^* - \left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i\right)\right)\right]}} \tag{24}$$

Also, the best reorder point and the optimal lot quantity are $s_i^* = -x_i^*(a_i - b_i p_i)\nu_i T^*$ and $Q_i^* = (a_i - b_i p_i)\nu_i T^*$ respectively.

Now for any $s_1^*, s_2^*, \ldots, s_N^*, T^*$, the first order condition to maximize the total profit function $\Pi(\vec{p}, \vec{s}^*, T^*)$ is

$$\frac{\partial \Pi(\vec{p}, \vec{s}^*, T^*)}{\partial p_i} = a_i \nu_i - 2b_i p_i \nu_i + b_i c_i \nu_i - \frac{b_i \nu_i T^*}{(\kappa_i + 1)}\left(\frac{h_i}{\alpha_i^{\kappa_i}} + w_i\right) + (h_i + w_i)\frac{b_i}{(\kappa_i + 1)\nu_i^{\kappa_i} T^{*\kappa_i}}\left[\frac{(s_i^* + (a_i - b_i p_i)\nu_i T^*)^{\kappa_i}((a_i - b_i p_i)\nu_i T^* - \kappa_i s_i^*)}{(a_i - b_i p_i)^{\kappa_i + 1}} - \frac{\kappa_i(-s_i^*)^{\kappa_i + 1}}{(\alpha_i - 1)^{\kappa_i}(a_i - b_i p_i)^{\kappa_i + 1}}\right] = 0 \tag{25}$$

Theorem 2: The total profit function $\Pi(\vec{p}, \vec{s}^*, T^*)$ is a concave function of $\vec{p}$ for a given $(\vec{s}^*, T^*)$.

Proof: Please see Appendix B.

4. Procedure for determining the optimal values of the model

In this section, with regard to that solving Eq. 25, using numerical methods takes a noticeable time and the optimized answers are hard to achieve, a simple procedure is presented to obtain good values for $\vec{\mathcal{p}}, \vec{s}, T$ and Π.

Step 1: For each product i; $\mathcal{p}_i = c_i$, where $c_i < \frac{a_i}{b_i}$. If

$$\frac{\sum_{i=1}^{N} \Lambda_i}{\sum_{i=1}^{N}[(\frac{(a_i-b_i\mathcal{p}_i)\nu_i}{(\kappa_i+1)})((\hbar_i + w_i)(1-x_i{}^*)^{\kappa_i} + \kappa_i w_i x_i{}^* - \left(\frac{\hbar_i}{\alpha_i{}^{\kappa_i}} + w_i\right))]} \geq 0,$$

calculate period length T^* and find out $\vec{s}^*$and $\Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*)$; otherwise, there is no feasible solution and then go to Step 5.

Step 2: For each product i $(i = 1, \dots, N)$, do: suppose that $\mathcal{p}_i = \mathcal{p}_i + \varepsilon$ and $\mathcal{p}_j = \mathcal{p}_j$, $\forall j \neq i$. Then, calculate the best reorder vector, the economic scheduling period and total profit function, and name them $(\vec{s}^*{}_i, T^*{}_i)$ and $\Pi_i(\vec{\mathcal{p}}, \vec{s}^*{}_i, T^*{}_i)$, respectively.

Step 3: Choose the product m that has the conditions below:

- $T^*{}_m$ can be calculated.
- $\mathcal{p}_m < \frac{a_m}{b_m}$.
- $\Pi_m(\vec{\mathcal{p}}, \vec{s}^*{}_m, T^*{}_m)$ is greater than the previous $\Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*)$ and greater than all $\Pi_j(\vec{\mathcal{p}}, \vec{s}^*{}_j, T^*{}_j), \forall j \neq m$.

If there is no product with these conditions, the best solutions are $\vec{s}^*, T^*, \Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*)$ and go to Step 5.

Step 4: $\mathcal{p}_m = \mathcal{p}_m + \varepsilon$, and $\mathcal{p}_j = \mathcal{p}_j, \forall j \neq m, T^* = T^*{}_m, \Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*) = \Pi_m(\vec{\mathcal{p}}, \vec{s}^*{}_m, T^*{}_m)$ and go to Step 2.

Step 5: End.

5. Results and discussion

In this section a numerical example is provided in order to illustrate proposed model and then numerical analyses are presented.

Example: Consider a production system with one item and the following values for the input parameters. $\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $a = 100$, $b = 2$, $c = 10$, $\alpha = 1.5$ and the index of the power demand pattern $\kappa = 3$. Now we have to solve equation $(1-x)^3 - \frac{x^3}{0.5^3} = \frac{5}{9}$. Using Newton method, $x^* = 0.161603$. Also $T^* = 0.0735$, $Q^* = 3438.5$, $s^* = -570.1081$, $\mathcal{p}^* = 30$ and $TP^* = 957280$. Also the graphical representation of $\Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*)$ is shown in Fig. 1 using input parameters of the example. Also according to the following values of parameters, Table 1 is provided: $\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $a = 100$, $b = 2$ and $\kappa = 3$. Optimum policies of the inventory system considering several combinations of parameters α and c are shown in Table 1.

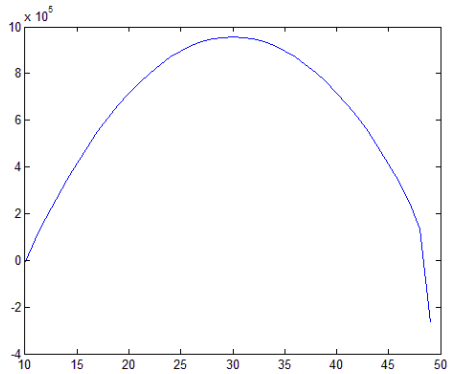

Fig. 1 Graphical representation of $\Pi(\vec{\mathcal{p}}, \vec{s}^*, T^*)$

Table 1 Optimal policies for the proposed EPQ model, considering several values for α and c

$\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $a = 100$, $b = 2$ and $\kappa = 3$							
Production rate	Production cost	x^*	T^*	s^*	Q^*	p^*	Π^*
$\alpha = 1.1$	$c = 10$	0.064156	0.1580	-486.6977	7394.1	30	958730
	$c = 15$		0.1714	-448.7131	6785.3	33	733230
	$c = 20$		0.1825	-421.4925	6347.0	35	538900
	$c = 25$		0.2040	-376.9944	5626.0	38	373420
	$c = 30$		0.2235	-344.1472	5089.3	40	239110
$\alpha = 1.3$	$c = 10$	0.135125	0.0900	-583.7937	4211.0	30	957780
	$c = 15$		0.0976	-538.2312	3864.3	33	732350
	$c = 20$		0.1039	-505.5802	3614.7	35	538080
	$c = 25$		0.1162	-452.2047	3204.1	38	372680
	$c = 30$		0.1273	-412.8045	2898.2	40	238430
$\alpha = 1.5$	$c = 10$	0.161603	0.0735	-570.1081	3438.5	30	957280
	$c = 15$		0.0797	-525.6137	3155.4	33	731890
	$c = 20$		0.0849	-493.7281	2951.6	35	537640
	$c = 25$		0.0949	-441.6038	2616.3	38	372290
	$c = 30$		0.1039	-403.1273	2366.5	40	238080
$\alpha = 1.7$	$c = 10$	0.170823	0.0666	-546.0560	3115.7	30	957000
	$c = 15$		0.0722	-503.4388	2859.1	33	731630
	$c = 20$		0.0769	-472.8984	2674.5	35	537400
	$c = 25$		0.0860	-422.9732	2370.7	38	372070
	$c = 30$		0.0942	-3861199	2144.4	40	237880
$\alpha = 1.9$	$c = 10$	0.174358	0.0630	-527.5052	2948.8	30	956830
	$c = 15$		0.0684	-486.3358	2706.0	33	731470
	$c = 20$		0.0728	-456.8329	2531.2	35	537250
	$c = 25$		0.0814	-408.6038	2243.7	38	371940
	$c = 30$		0.0891	-373.0025	2029.5	40	237760

Table 2 shows the optimal policies of the system considering combinations of parameters κ and c using following parameters: $\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $a = 100$, $b = 2$ and $\alpha = 1.5$.

Table 3 shows the optimal policies of the system considering different values of a using following parameters: $\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $b = 2$, $c = 10$, $\kappa = 4$ and $\alpha = 1.7$.

Table 2 Optimal policies for the proposed EPQ model, considering several values for κ and c

$\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $a = 100$, $b = 2$ and $\alpha = 1.5$							
Production rate	Production cost	x^*	T^*	s^*	Q^*	p^*	Π^*
$\kappa = 0.5$	$c = 10$	0.081199	0.0744	-290.1504	3482.8	30	957310
	$c = 20$		0.0860	-251.2777	2989.7	35	537670
$\kappa = 1$	$c = 10$	0.148148	0.0750	-533.3328	3508.8	30	957330
	$c = 20$		0.0866	-461.8791	3012.0	35	537690
$\kappa = 2$	$c = 10$	0.175842	0.0756	-638.0297	3536.6	30	957350
	$c = 20$		0.0873	-522.5499	3035.8	35	537710
$\kappa = 3$	$c = 10$	0.161603	0.0735	-570.1081	3438.5	30	957280
	$c = 20$		0.0849	-493.7281	2951.6	35	537640

Table 3 Optimal policies for the proposed EPQ model, considering different values of a

$\Lambda = 100$, $\hbar = 4$, $w = 5$, $\nu = 1200$, $b = 2$, $c = 10$, $\kappa = 4$ and $\alpha = 1.7$						
a	x^*	T^*	s^*	Q^*	p^*	Π^*
$a = 100$	0.170823	0.0666	-546.0560	3115.7	30	957000
$a = 200$		0.0444	-819.0840	4741.4	30	4855500
$a = 300$		0.0356	-1021.6	5937.5	30	11754000

Table 4 Optimal policies for the proposed EPQ model, considering different values of b

$\Lambda = 100$, $\hbar = 4$, $\mathscr{w} = 5$, $\nu = 1200$, $a = 100$, $c = 10$, $\kappa = 4$ and $\alpha = 1.7$						
b	x^*	T^*	s^*	Q^*	$\mathscr{p}^*$	Π^*
$b = 1$	0.170823	0.0628	-579.1799	3352.6	55	2426800
$b = 2$		0.0666	-546.0560	3115.7	30	957000
$b = 3$		0.0722	-503.4388	2814.1	22	486130

Table 4 shows the optimal policies of the system considering different values of b using following parameters: $\Lambda = 100$, $\hbar = 4$, $\mathscr{w} = 5$, $\nu = 1200$, $a = 100$, $c = 10$, $\kappa = 4$ and $\alpha = 1.7$.

Figs. 2 and 3 show the cycle length and the total profit as functions of unit production cost, when input parameters are used from Table 1. In each figure different values of α are considered ($\alpha = 1.1, 1.3, 1.5, 1.9$).

Figs. 4 and 5 show changes of the lot size and total profit respect to the changes of the index of demand pattern using Table 2. In each figure two values of production cost c are considered ($c = 10, 20$).

Figs. 6 and 7 show changes of the price respect to the changes of the parameters a and b using Table 3 and 4, respectively.

Some managerial insights can be expressed as follows.

In Table 1, by fixing the replenishment rate parameter α, if the unit production cost c increases then the total profit function Π^*, the best lot size Q^* and the value of s^* decrease. However, the optimal cycle length T^* and the optimum price $\mathscr{p}^*$ increase in the same situation.

In the same Table 1, fixed the unit production cost c, the total profit function Π^*, slightly, the best scheduling period T^* and the economic lot quantity Q^* decrease as the production rate α increases. However, In the same conditions the optimal price $\mathscr{p}^*$ does not change.

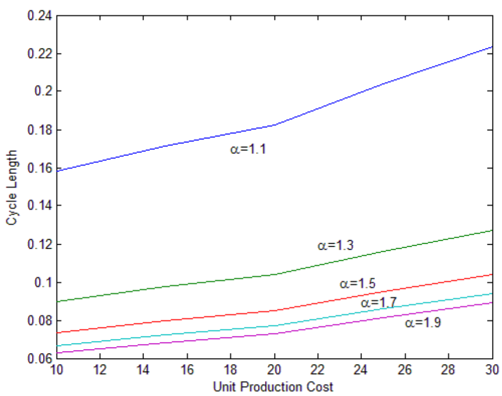

Fig. 2 Changes of cycle length respect to the changes of unit production cost using Table 1

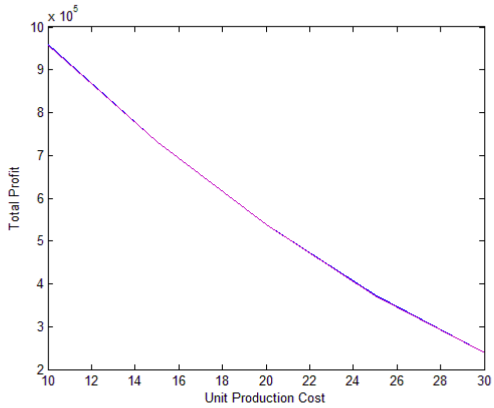

Fig. 3 Changes of total profit respect to the changes of unit production cost using Table 1

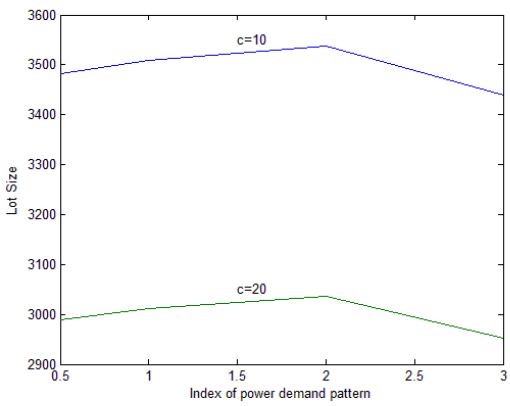

Fig. 4 Changes of lot size respect to the changes of κ using Table 2

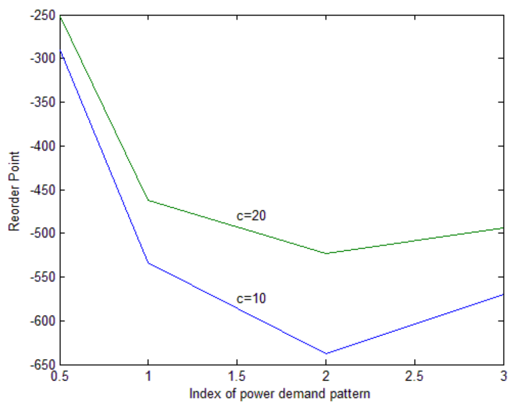

Fig. 5 Changes of reorder point respect to the changes of κ using Table 2

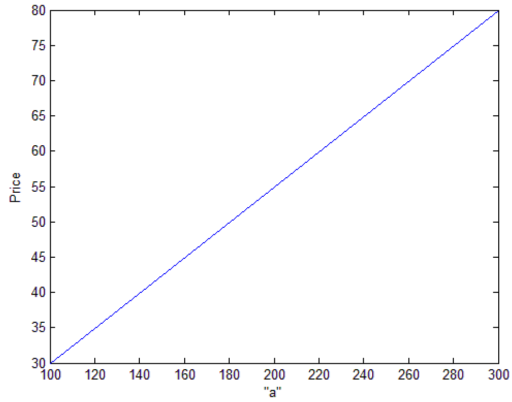

Fig. 6 Changes of price respect to the changes of parameter a using Table 3

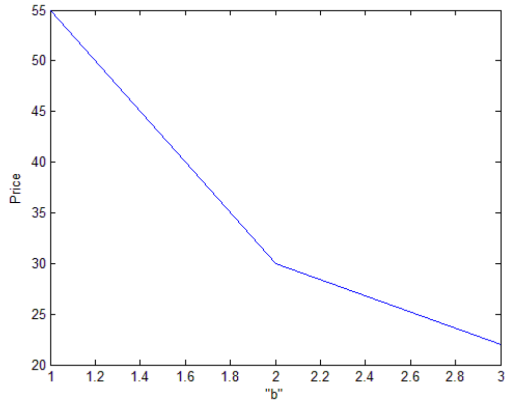

Fig. 7 Changes of price respect to the changes of parameter b using Table 4

Using Table 2, by fixing the index of power demand κ unit production cost c, if the unit production cost c increases then the total profit function Π^*, the economic lot quantity Q^* and the value of s^*decrease. In the same conditions the best price $\wp^*$ and the optimum cycle length T^* increase.

In Table 3, fixing c, α, κ and b, if parameter a increases then the total profit function Π^*, the price $\wp^*$ and the economic lot quantity Q^* increase. However the optimal scheduling period T^* decreases in the same conditions.

In Table 4, fixing c, α, κ and a, if parameter b increases then the total profit function Π^*, the best price $\wp^*$ and the economic lot quantity Q^* decrease. However the optimum scheduling period T^* increases in the same conditions.

6. Conclusions and future research

In this paper, an economic production model has been presented using kind of demand rate named power demand. Also it is supposed that demand of customers dependes on price linearly and production rate changes pro rata with demand rate. Multiple products are assumed to be in the inventory system and shortages are allowed and fully backlogged. Mathematical modeling and optimization methods are used to find optimal solutions for both single-product and multiple-products situations. Also since that achieving optimum inventory policies for second situation is hard a simple heuristic procedure is proposed to obtain the near-optimal solutions for multi-items form. Several examples are presented to illustrate the applications of the model using various values of parameters. The outcomes reveal that fixing the production rate parameter, if the unit production cost increases then the total profit function and the economic lot size decrease, while the best cycle length and the optimum price increase. The total profit function slightly, the best scheduling period and the optimal lot quantity decrease as the production rate increases. By fixing the index of power demand, if the unit production cost increases, the total profit function and the economic lot quantity decrease while the best price and the optimal cycle length increase. In the end, the following suggestions for further research are made:

- The proposed model can be extended using deterioration in the inventory system.
- Allowing for shortages that are lost sales or partially backorderd could be considered.
- Imperfect items are produced in many production systems as well as perfect items. So it could be considered in the inventory system.
- Since that different price functions may be possible in the real world situations, they could be considered in the inventory system.
- The proposed model can be considered with manufacturing disruption costs.
- Taking sustainability concerns into account could be another interesting recommendation.

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Appendix A

For any given $\vec{p}$, The total profit function $\Pi(\vec{p},\vec{s},T)$ is concave.

Proof: The Hessian matrix can be used to prove the concavity of $\Pi(\vec{p},\vec{s},T)$.

$$\begin{aligned}\Pi(\vec{p},\vec{s},T) &= \sum_{i=1}^{N} TP_i(p_i,s_i,T) = \sum_{i=1}^{N}\Big[(a_i-b_ip_i)p_i\nu_i - \frac{(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i+1}(h_i+w_i)}{(\kappa_i+1)(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}}\\ &-\frac{(-s_i)^{\kappa_i+1}(h_i+w_i)}{(\kappa_i+1)(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}} + \frac{(a_i-b_ip_i)\nu_iT}{(\kappa_i+1)}\left(\frac{h_i}{\alpha_i^{\kappa_i}}+w_i\right)+w_is_i-\frac{\Lambda_i}{T}\\ &-(a_i-b_ip_i)c_i\nu_i\Big]\end{aligned} \tag{A1}$$

$$\frac{\partial\Pi(\vec{p},\vec{s},T)}{\partial s_i} = \frac{(-s_i)^{\kappa_i}(h_i+w_i)}{(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}} - \frac{(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i}(h_i+w_i)}{(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}} + w_i,\ \ \forall i \tag{A2}$$

$$\begin{aligned}\frac{\partial\Pi(\vec{p},\vec{s},T)}{\partial T} &= \sum_{i=1}^{N}\frac{\kappa_i(-s_i)^{\kappa_i+1}(h_i+w_i)}{(\kappa_i+1)(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+1}}\\ &-\sum_{i=1}^{N}\frac{(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i}((a_i-b_ip_i)\nu_iT-\kappa_is_i)(h_i+w_i)}{(\kappa_i+1)(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+1}}\\ &+\sum_{i=1}^{N}\frac{(a_i-b_ip_i)\nu_i}{(\kappa_i+1)}\left(\frac{h_i}{\alpha_i^{\kappa_i}}+w_i\right)+\frac{\sum_{i=1}^{N}\Lambda_i}{T^2}\end{aligned} \tag{A3}$$

$$\frac{\partial^2\Pi(\vec{p},\vec{s},T)}{\partial s_i^2} = -\frac{\kappa_i(-s_i)^{\kappa_i-1}(h_i+w_i)}{(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}} - \frac{\kappa_i(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i-1}(h_i+w_i)}{(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i}} < 0,\ \ \forall i \tag{A4}$$

$$\begin{aligned}\frac{\partial^2\Pi(\vec{p},\vec{s},T)}{\partial T^2} &= -\sum_{i=1}^{N}\frac{\kappa_i(-s_i)^{\kappa_i+1}(h_i+w_i)}{(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+2}}\\ &-\sum_{i=1}^{N}\frac{\kappa_is_i^2(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i-1}(h_i+w_i)}{(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+2}} - \frac{2\sum_{i=1}^{N}\Lambda_i}{T^3} < 0\end{aligned} \tag{A5}$$

$$\frac{\partial^2\Pi(\vec{p},\vec{s},T)}{\partial s_i\partial T} = -\frac{\kappa_i(-s_i)^{\kappa_i}(h_i+w_i)}{(\alpha_i-1)^{\kappa_i}(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+1}} + \frac{\kappa_is_i(s_i+(a_i-b_ip_i)\nu_iT)^{\kappa_i-1}(h_i+w_i)}{(a_i-b_ip_i)^{\kappa_i}\nu_i^{\kappa_i}T^{\kappa_i+1}},\ \ \forall i \tag{A6}$$

$$\frac{\partial^2 \Pi(\vec{\boldsymbol{\wp}},\vec{\boldsymbol{s}},T)}{\partial s_i \partial s_j} = 0, \quad \forall i,j \tag{A7}$$

We suppose that $s_i + (a_i - b_i \wp_i)\nu_i T \geq 0$.

$$[T, s_1, s_2, \dots, s_N] \begin{bmatrix} \frac{\partial^2 \Pi}{\partial T^2} & \frac{\partial^2 \Pi}{\partial T \partial s_1} & \cdots & \frac{\partial^2 \Pi}{\partial T \partial s_N} \\ \frac{\partial^2 \Pi}{\partial s_1 \partial T} & \frac{\partial^2 \Pi}{\partial {s_1}^2} & \cdots & \frac{\partial^2 \Pi}{\partial s_1 \partial s_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 \Pi}{\partial s_N \partial T} & \frac{\partial^2 \Pi}{\partial s_N \partial s_1} & \cdots & \frac{\partial^2 \Pi}{\partial {s_N}^2} \end{bmatrix} \begin{bmatrix} T \\ s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} = \tag{A8}$$

$$= T^2 \frac{\partial^2 \Pi}{\partial T^2} + {s_1}^2 \frac{\partial^2 \Pi}{\partial {s_1}^2} + \cdots + {s_N}^2 \frac{\partial^2 \Pi}{\partial {s_N}^2} + 2T \left(s_1 \frac{\partial^2 \Pi}{\partial s_1 \partial T} + \cdots + s_N \frac{\partial^2 \Pi}{\partial s_N \partial T} \right) = \frac{2 \sum_{i=1}^{N} \Lambda_i}{T^3} < 0$$

Appendix B

The total profit function $\Pi(\vec{\boldsymbol{\wp}}, \vec{\boldsymbol{s}}^*, T^*)$ is a concave function of $\vec{\boldsymbol{\wp}}$ for a given $(\vec{\boldsymbol{s}}^*, T^*)$.

Proof: We have

$$\frac{\partial \Pi(\vec{\wp}, \vec{s}^*, T^*)}{\partial \wp_i} = a_i \nu_i - 2 b_i \wp_i \nu_i + b_i c_i \nu_i - \frac{b_i \nu_i T^*}{(\kappa_i + 1)} \left(\frac{\hbar_i}{{\alpha_i}^{\kappa_i}} + w_i \right) + \frac{b_i(\hbar_i + w_i)}{(\kappa_i + 1){\nu_i}^{\kappa_i} T^{*\kappa_i}} \left[\frac{({s_i}^* + (a_i - b_i \wp_i)\nu_i T^*)^{\kappa_i}((a_i - b_i \wp_i)\nu_i T^* - \kappa_i {s_i}^*)}{(a_i - b_i \wp_i)^{\kappa_i + 1}} - \frac{\kappa_i (-{s_i}^*)^{\kappa_i + 1}}{(\alpha_i - 1)^{\kappa_i}(a_i - b_i \wp_i)^{\kappa_i + 1}} \right] \tag{B1}$$

$$\frac{\partial^2 \Pi(\vec{\wp}, \vec{s}^*, T^*)}{\partial {\wp_i}^2} = -2 b_i \nu_i - \frac{\kappa_i {b_i}^2 (-{s_i}^*)^{\kappa_i + 1}(\hbar_i + w_i)}{(\alpha_i - 1)^{\kappa_i} {\nu_i}^{\kappa_i} T^{*\kappa_i} (a_i - b_i \wp_i)^{\kappa_i + 2}} - \frac{\kappa_i {b_i}^2 {s_i}^{*2} ({s_i}^* + (a_i - b_i \wp_i)\nu_i T^*)^{\kappa_i - 1}(\hbar_i + w_i)}{{\nu_i}^{\kappa_i} T^{*\kappa_i} (a_i - b_i \wp_i)^{\kappa_i + 2}} < 0 \tag{B2}$$

$$\frac{\partial^2 \Pi(\vec{\wp}, \vec{s}^*, T^*)}{\partial \wp_i \partial \wp_j} = 0 \tag{B3}$$

And so that

$$[\wp_1, \wp_2, \dots, \wp_N] \begin{bmatrix} \frac{\partial^2 \Pi}{\partial {\wp_1}^2} & \frac{\partial^2 \Pi}{\partial \wp_1 \partial \wp_2} & \cdots & \frac{\partial^2 \Pi}{\partial \wp_1 \partial \wp_N} \\ \frac{\partial^2 \Pi}{\partial \wp_2 \partial \wp_1} & \frac{\partial^2 \Pi}{\partial {\wp_2}^2} & \cdots & \frac{\partial^2 \Pi}{\partial \wp_2 \partial \wp_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 \Pi}{\partial \wp_N \partial \wp_1} & \frac{\partial^2 \Pi}{\partial \wp_N \partial \wp_2} & \cdots & \frac{\partial^2 \Pi}{\partial {\wp_N}^2} \end{bmatrix} \begin{bmatrix} \wp_1 \\ \wp_2 \\ \vdots \\ \wp_N \end{bmatrix} = \sum_{i=1}^{N} {\wp_i}^2 \frac{\partial^2 \Pi(\vec{\wp}, \vec{s}^*, T^*)}{\partial {\wp_i}^2} < 0 \tag{B4}$$